Optimization

Homework 1 Solutions

1.

(a). We can rewrite f as

$$f(x) = \frac{1}{2}\ x^T \begin{bmatrix} 2 & 6 \\ 6 & 14 \end{bmatrix} x + x^T \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6$$

The gradient and Hessian of f are

$$\nabla f(x) = \begin{bmatrix} 2 & 6 \\ 6 & 14 \end{bmatrix} x + \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

$$F(x) = \begin{bmatrix} 2 & 6 \\ 6 & 14 \end{bmatrix}.$$

Hence $\nabla f([1,1]^T) = [11,25]^T$.

(b). Suppose $d = [d_1, d_2]^T$ and $d_1^2 + d_2^2 = 1$

Then directional derivative = $\nabla f(x^T)\,d = [11,25]\begin{bmatrix} d1 \\ d2 \end{bmatrix} = 11\,d_1 + 25\,d_2$

Let $f = 11\,d_1 + 25\,(1 - d_1^2)^{1/2}$

$f' = 11 + \frac{25}{2}(1 - d_1^2)^{-1/2}(-2\,d_1) = 0$

$d_1 = \frac{11}{\sqrt{11^2+25^2}}$, $d_2^2 = \frac{25}{\sqrt{11^2+25^2}}$. Hence, $d = [\frac{11}{\sqrt{11^2+25^2}}, \frac{25}{\sqrt{11^2+25^2}}]^T$.

(c). The FONC in this case is $\nabla f(x) = 0$, The only point satisfying the FONC is

$x^* = \frac{1}{2}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$,

The point above does not satisfy the SONC, because the Hessian is not positive definite (its determinant is negative). Therefore, f does not have a minimizer.

2. .

(a). We can rewrite f as

$$f(x) = \frac{1}{2}\ x^T \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix} x + x^T \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7$$

The gradient and Hessian of f are

$$\nabla f(x) = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

$$F(x) = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix}.$$

Hence $\bigtriangledown f([0,1]^T) = [7,6]^T$.

directional derivative = $\bigtriangledown f(x^T)$ d= 7

(b). The FONC in this case is $\bigtriangledown f(x) = 0$, The only point satisfying the FONC is

$x^* = \frac{1}{4}\begin{bmatrix}-5\\2\end{bmatrix}$,*f* does not have a minimizer, because it does not satisfy SONC.

3. $\bigtriangledown f(x) = [0,5]^T$, $d^T\bigtriangledown f(x) = 6\, d_2$, where $d = [d_1, d_2]^T$

(a). Because d_2 is allowed to be less than zero, it does not satisfy FONC.

(b). It satisfies SONC.

(c). x^* is not a local minimizer.

4. .

(a). x^* satisfies FONC.

(b). x^* does not satisfy SONC. ex: $F(x) = \begin{bmatrix}8 & 0\\0 & -2\end{bmatrix}$, if $d_1 = \begin{bmatrix}1\\0\end{bmatrix}$, then $d_1^T F\, d_1 > 0$

if $d_2 = \begin{bmatrix}0\\1\end{bmatrix}$, then $d_2^T F\, d_2 < 0$

(c). x^* is not a local minimizer.

5. $f'(x) = \bigtriangledown f(x) = 2x - 4\sin x$.

$f''(x) = F(x) = 2 - 4\cos x$.

Using Newton's method： $x^{(k+1)} = x^{(k)} - \frac{f'(x)}{f''(x)}$.

$x^{(1)} = -7.4727$; $x^{(2)} = 14.4785$; $x^{(3)} = 6.9351$; $x^{(4)} = 16.6354$.

6. $g'(x) = \bigtriangledown f(x) = 4(2x-1)+2^{14}(4-1024x)^3$

Using $x^{(k)} = [g(x^{(k)})\, x^{(k-1)} - g(x^{(k-1)})\, x^{(k)}]/[\, g(x^{(k)}) - g(x^{(k-1)})\,]$ to calculate

until $| x^{(18)} - x^{(17)} | < x^{(17)} * 10^{-5}$

Hence, x^* is 0.0039671

$g(x^*) = 0.9846$.

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